

ON WEYL'S EMBEDDING PROBLEM IN RIEMANNIAN MANIFOLDS

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ABSTRACT. We consider a priori estimates of the Weyl's embedding problem of (\mathbb{S}^2, g) in general 3-dimensional Riemannian manifold (N^3, \bar{g}) . We establish the mean curvature estimate under natural geometric assumption. Together with a recent work by Li-Wang, we obtain an isometric embedding of (\mathbb{S}^2, g) in Riemannian manifold. In addition, we reprove Weyl's isometric embedding theorem in space form under the condition that $g \in C^2$ with D^2g Dini continuous.

1. INTRODUCTION

Weyl's embedding problem is a classic isometric embedding problem in differential geometry. It was raised by Weyl [37] in 1916. The problem concerns the realization of (\mathbb{S}^2, g) with positive Gauss curvature as a convex surface in 3-dimensional Euclidean space.

In the case that g is analytic, it was solved by Lewy in [19]. In the smooth case, Weyl's problem was fully resolved by Nirenberg in a milestone paper [27], under conditions that $g \in C^4$ and Gauss curvature $K_g > 0$. In order to reduce the regularity assumption on g , Heinz [12, 14] developed the interior C^2 estimate for the embedding. This allows him to solve the problem under the condition $g \in C^3$. This method was further modified by Schulz to relax the condition to $g \in C^{2,\alpha}$, see in [30]. In the degenerate case $K_g \geq 0$, Weyl's problem was studied by Guan-Li [9] and Hong-Zuily [16], see also [18].

In the case of hyperbolic space \mathbb{H}^3 , Weyl's problem was solved by Pogorelov [28] under conditions that $g \in C^4$ and $K_g > -1$. The degenerate case $K_g \geq -1$ was studied by Lin-Wang [22] recently, see also early results by Chang-Xiao [5].

For general ambient space other than space form, Pogorelov established isometric embedding of a convex surface (\mathbb{S}^2, g) into (N, \bar{g}) under certain conditions on sectional curvatures of the ambient space, see [29] for details, see also work by Dubrovin [6]. In a recent paper, Guan and the author [10] obtained a mean curvature bound if (M^n, g) is isometrically embedded into warped product space and a class of ambient space. On the other hand, Heinz [15] obtained a mean curvature estimate of a convex surface (M^2, g) isometrically embedded into (N, \bar{g}) under conditions that $g \in C^3$ and the existence of a global convex function in ambient space.

Another motivation to study Weyl's problem comes from general relativity. This can be seen by the definitions of quasi-local mass. The question concerns how to

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define the mass of a bounded domain (Ω, \bar{g}) in space time. Suppose $\Sigma = \partial\Omega$ is a topological sphere with positive Gauss curvature. In the time symmetric case, the Brown-York mass [4] is defined to be:

$$m_{BY} = \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) d\sigma,$$

where H is the mean curvature of Σ in (Ω, \bar{g}) and H_0 is the mean curvature of Σ embedded into \mathbb{R}^3 . While in the case of space time, the Liu-Yau quasi-local mass [23, 24] is defined to be

$$m_{LY} = \frac{1}{8\pi} \int_{\Sigma} (H_0 - |H|) d\sigma,$$

where $|H|$ is the Lorentzian norm of the mean curvature vector. The positivity of m_{BY} was established by Shi-Tam [33], the positivity of m_{LY} was established by Liu-Yau [24]. By the above two definitions, we see that the solution of isometric embedding in \mathbb{R}^3 plays a crucial role.

In the case that the Gauss curvature of Σ ceases to be positive, Wang-Yau [35] generalized the Liu-Yau quasi-local mass, using Pogorelov's solution to Weyl's problem in hyperbolic space $\mathbb{H}_{\kappa^2}^3$. Their result was later furnished by Shi-Tam [34] to obtain the following inequality,

$$\int_{\Sigma} (H_0 - H) \cosh(\kappa r) d\sigma \geq 0,$$

where H_0 is the mean curvature of Σ embedded into $\mathbb{H}_{\kappa^2}^3$ and r is the distance function in $\mathbb{H}_{\kappa^2}^3$. It turns out that this inequality plays an important role in our mean curvature estimate. In space time case, Wang-Yau [36] further studied a new quasi-local mass using isometric embedding into Minkowski spacetime.

In general relativity, other than space form, a more interesting model is the Schwarzschild manifold. This can be seen from the Riemannian Penrose inequality [3, 17], the equality holds if and only if the ambient space is Schwarzschild manifold. Moreover, if we can isometric embed (\mathbb{S}^2, g) into Schwarzschild manifold such that it's homologous to the horizon, then Miao [26] was able to define a local mass and prove a localized Riemannian Penrose inequality. Thus the solution of isometric embedding into Schwarzschild manifold is of significant importance to help understand not only the quasi-local mass, but also the localized Riemannian Penrose inequality.

In this paper, we consider a priori estimates of isometrically embedded surfaces (\mathbb{S}^2, g) in 3-dimensional Riemannian manifolds (N^3, \bar{g}) . Since the ambient space is no longer a homogeneous space, thus the C^0 a priori estimate can not be automatically obtained by the metric as (\mathbb{S}^2, g) may drift to infinity. Therefore, one need to study carefully on the a priori estimate of the linearized equation. This was carried out in a recent work [20] by Li-Wang, they were able to obtain the uniform C^0 estimate and openness under the condition that the mean curvature is uniformly bounded, in particular the bound should be independent of the position of (\mathbb{S}^2, g) in (N, \bar{g}) . However, the estimate obtained in [15] and [10] does depend on the position of (\mathbb{S}^2, g) , thus may not serve the purpose.

We adopt Heinz's interior C^2 estimate to attack the problem. There're two major advantages of this method. The first is that our mean curvature estimate is indeed independent of the position of (\mathbb{S}^2, g) in (N, \bar{g}) . The second is that we only require the metric g and \bar{g} up to third derivatives, where both $g, \bar{g} \in C^4$ is required in [10].

A key observation in our proof is a bound of the total mean curvature. In Euclidean space, the total mean curvature can be bounded via the classic Minkowski formula, which was carried out by Heinz in [13], see also [32]. Using the same idea, we're able to employ the analogous Minkowski formula to bound the total mean curvature in the case of space form. However, for general Riemannian manifold, in lack of conformal Killing vector field, the Minkowski formula no longer exists. In order to bound the total mean curvature, we use the positivity of hyperbolic version of quasi-local mass, which is proved by Wang-Yau [35] and Shi-Tam [34]. The total mean curvature is thus bounded due to the estimate of isometric embedding in hyperbolic space.

Before we state our theorem, let's fix some notation. Let (\mathbb{S}^2, g) be an isometrically embedded surface in an ambient space (N^3, \bar{g}) . Denote Ric and \bar{Ric} the Ricci curvature tensors of (\mathbb{S}^2, g) and (N, \bar{g}) respectively, and denote $R = Tr(Ric)$ and $\bar{R} = Tr(\bar{Ric})$ the scalar curvatures of (\mathbb{S}^2, g) and (N, \bar{g}) respectively. Let ν be the unit outer normal, and h the corresponding second fundamental form, by Gauss equation, we have

$$(1.1) \quad \sigma_2(h) = \frac{R - \bar{R}}{2} + \bar{Ric}(\nu, \nu).$$

We now state our main results.

Theorem 1.1. *Let (N^3, \bar{g}) be a 3-dimensional Riemannian manifold with possibly finite many boundary components, which are minimal surfaces. Suppose $\Sigma = (\mathbb{S}^2, g)$ is isometrically embedded into (N^3, \bar{g}) , let X be the embedding, assume that*

$$(1.2) \quad R(x) - \bar{R}(X(x)) + 2\bar{Ric}_{X(x)}(\nu, \nu) \geq C_0 > 0,$$

$\forall x \in \Sigma$. Assume further that $\bar{R} \geq -6\kappa^2$ for some constant $\kappa > 0$. Then we have

$$\|D^2 X\|_{C^{k, \mu}} \leq C$$

for any $0 < \mu < 1$, where C depends only on $C_0, \kappa, \|g\|_{C^{k+3}}$ and $\|\bar{g}\|_{C^{k+3}}$, but not on the position of Σ in N , here $k \geq 0$ is an integer.

By Gauss equation (1.1), assumption (1.2) implies $\sigma_2(h) \geq C_0$, which is the ellipticity condition in the view of partial differential equations. In particular, the mean curvature estimate corresponds to the case $k = 0$, we refer to Theorem 4.2 in section 4 for detail.

Together with the recent work by Li-Wang [20] (the openness part of the isometric embedding problem), we can prove an isometric embedding in Riemannian manifold. In particular, we are able to prove an isometric embedding in warped product space without horizon or outside the horizon.

Theorem 1.2. *Suppose (N^3, \bar{g}) is a C^3 Riemannian manifold with possibly finite many boundary components, which are minimal surfaces. Suppose that the scalar curvature \bar{R} is bounded below, let g be a C^3 metric on \mathbb{S}^2 such that*

$$(1.3) \quad R(x) \geq \bar{R}(X) - 2 \inf\{\bar{Ric}(X)(\xi, \xi) | \xi \in T_X N, |\xi| = 1\} + C_0,$$

$\forall x \in (\mathbb{S}^2, g)$ and $\forall X \in (N, \bar{g})$, where $C_0 > 0$ is a constant. Then (\mathbb{S}^2, g) can be isometrically embedded into (N, \bar{g}) as a $C^{2, \mu}$ surface which is null-homologous to the boundary component, for any $0 < \mu < 1$. Moreover, if $g, \bar{g} \in C^k$, $k \geq 3$, then the embedding is $C^{k-1, \mu}$. In particular, if both g, \bar{g} are smooth, then the embedding is also smooth.

This Theorem is also presented and proved by Li-Wang in [20] and Guan-Lu in [10].

In view of Theorem 1.1 and Theorem 1.2, we require $g \in C^3$ to establish C^2 embedding. Recall that in the case of Euclidean space, Schulz was able to establish the embedding under the condition that $g \in C^{2, \alpha}$. A natural question is what's the optimal regularity condition to ensure C^2 embedding. In general, a weaker regularity condition to ensure corresponding regularity is the Dini continuity for elliptic partial differential equations. We say a function is Dini continuous if the moduli of continuity $\omega(r)$ satisfies

$$\int_0^1 \frac{\omega(r)}{r} dr < \infty.$$

It's easy to see that Hölder continuity implies Dini continuity.

In this direction, we are able to weaken the condition to $g \in C^2$ with D^2g Dini continuous in the case of space form.

Theorem 1.3. *Let g be a C^2 metric on \mathbb{S}^2 , with D^2g Dini continuous. Suppose the scalar curvature of (\mathbb{S}^2, g) satisfies*

$$R > 2K,$$

then (\mathbb{S}^2, g) can be isometrically embedded into N_K^3 as a C^2 surface Σ , where N_K^3 denotes 3-dimensional space form with constant sectional curvature K .

By C^2 surface, we mean that the embedding X is C^2 , with moduli of continuity of D^2X under control. This is different from $C^{1,1}$ embedding. We remark that this theorem is new even in Euclidean space.

The organization of the paper is as follows. In next section, we state some lemmas concerning the classic isothermal parameters, quasi-local mass and Heinz's interior C^2 estimate. In section 3, we consider the isometric embedding in space form. In section 4, we consider the a priori estimates of the embedding and prove Theorem 1.1. We study the isometric embedding into Riemannian manifold in section 5. The proof of technical Lemma 2.6 will be given in the Appendix.

2. PRELIMINARY

Let's list some basic formulas for isometrically immersed surface (M^2, g) in an ambient Riemannian manifold (N^3, \bar{g}) . Denote R_{ijkl} and \bar{R}_{abcd} to be the Riemannian curvatures of M and N respectively. For a fixed local frame (e_1, e_2) on M , let ν be a normal vector field of M , and let $h = (h_{ij})$ be the second fundamental form of M with respect to ν . We have the Gauss equation and Codazzi equation,

$$(2.1) \quad R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \quad (\text{Gauss})$$

$$(2.2) \quad \nabla_k h_{ij} = \nabla_j h_{ik} + \bar{R}_{\nu ijk}. \quad (\text{Codazzi})$$

We use the convention that R_{ijij} denotes the sectional curvature.

Taking trace of (2.1), we have

$$(2.3) \quad \det(h_j^i) = \frac{R - \bar{R}}{2} + \bar{R}ic(\nu, \nu)$$

where $h_j^i = g^{ik}h_{kj}$ is the Weingarten tensor.

We first state a classic result about the existence of isothermal parameters of surface by Hartman-Wintner, see in [11].

Lemma 2.1. *Let Σ be a surface and B_r a domain in Σ . Let (l_{ij}) be a 2×2 , positive definite symmetric matrix in B_r , such that l_{ij} is Dini continuous in B_r . Then there exist isothermal parameters (u, v) such that*

$$ds^2 = l_{11}dx^2 + 2l_{12}dxdy + l_{22}dy^2 = \Lambda(du^2 + dv^2), \quad \Lambda \neq 0$$

in B_r for r sufficiently small.

We also need the following lemma concerning the positivity of quasi-local mass, which was proved by Wang-Yau [35] and Shi-Tam [34].

Lemma 2.2. *Let (Ω, \bar{g}) be a 3-dimentional Riemannian manifold with scalar curvature $\bar{R} \geq -6\kappa^2$ for some $\kappa > 0$. Assume that Σ is the boundary of (Ω, \bar{g}) , which is a topological sphere with scalar curvature $R > -2\kappa^2$ and positive mean curvature H . Then Σ can be isometrically embedded into $\mathbb{H}_{\kappa^2}^3$, let H_0 be the corresponding mean curvature, then we have*

$$\int_{\Sigma} (H_0 - H) \cosh(\kappa r) d\sigma \geq 0$$

where r is the distance function from the origin in $\mathbb{H}_{\kappa^2}^3$.

In the case that (N^3, \bar{g}) has horizons, we need extend Lemma 2.2 to the case that (Ω, \bar{g}) has more than one boundary components. This is proved in a recent work by Mantoulidis-Miao [25] based on the work of Wang-Yau [35] and Shi-Tam [34].

Lemma 2.3. *Let (Ω, \bar{g}) be a 3-dimentional Riemannian manifold with scalar curvature $\bar{R} \geq -6\kappa^2$ for some $\kappa > 0$. Assume that Σ is one conneted component of the boundary of (Ω, \bar{g}) , which is a topological sphere with scalar curvature $R > -2\kappa^2$ and positive mean curvature H . Assume further that all other connected components of*

the boundary of (Ω, \bar{g}) are minimal surfaces. Then Σ can be isometrically embedded into $\mathbb{H}_{\kappa^2}^3$, let H_0 be the corresponding mean curvature, then we have

$$\int_{\Sigma} (H_0 - H) \cosh(\kappa r) d\sigma \geq 0$$

where r is the distance function from the origin in $\mathbb{H}_{\kappa^2}^3$.

Remark 2.4. We remark that the assumption in [25] is less restrictive than Lemma 2.3, in fact as long as the mean curvature of all other boundary component of (Ω, \bar{g}) w.r.t. the outer normal is greater or equal than -2κ , the conclusion still holds. In our case, Lemma 2.3 is suffice as we already assume all boundary component of (N, \bar{g}) are minimal surfaces.

We attack the problem using Heinz's interior C^2 estimate. The procedure involves two steps. The first step is to establish a suitable equation system and to transform the problem to the estimate of the elliptic system. The second to obtain a lower bound of the Jacobian $J(x, y)$ of the map. The following lemma is in fact the second step, see Theorem 8.3.2 in [30].

Consider the following elliptic systems of Heinz-Lewy type, let $z = (x, y) = z(u, v) = z(w) \in W^{1,2}(\Omega)$ satisfying,

$$(2.4) \quad \begin{aligned} Lx &= h_1(z)|Dx|^2 + h_2(z)Dx \cdot Dy + h_3(z)|Dy|^2 + h_4(z)Dx \wedge Dy, \\ Ly &= \tilde{h}_1(z)|Dx|^2 + \tilde{h}_2(z)Dx \cdot Dy + \tilde{h}_3(z)|Dy|^2 + \tilde{h}_4(z)Dx \wedge Dy, \end{aligned}$$

where

$$L = -\frac{1}{a(z, w)} D_{\alpha} (a(z, w) D_{\alpha}), \quad Du \wedge Dv = u_x v_y - u_y v_x$$

Assumption:

(1) $a \in C^{\mu}(\Omega \times \mathbb{R}^2)$, such that

$$0 < \lambda \leq a(z, w) \leq \Lambda < +\infty, \quad [a]_{\mu} \leq M_1$$

(2)

$$|h_1(w)|, \dots, |\tilde{h}_4(w)| \leq M_0$$

(3)

$$\tilde{h}_1(w) = h_1(w) - \tilde{h}_2(w) = h_2(w) - \tilde{h}_3(w) = h_3(w) = 0.$$

(4) a is Dini continuous with moduli of continuity ω , such that

$$0 < \lambda \leq a(z, w) \leq \Lambda < +\infty,$$

(5)

$$h_1(w) = \dots = \tilde{h}_4(w) = 0.$$

Lemma 2.5. *Let $z(w)$ be a homeomorphism from the closed unit disc \bar{B} onto itself of class $W^{1,2}$ solving the system (2.4). Suppose that assumption (1,2,3) is satisfied, assume that $z(0) = 0$ and*

$$\int_{\Omega} |Dz|^2 dudv \leq M$$

Then $z \in C_{loc}^{1,\mu}(B) \cap C^0(\bar{B})$ and the Jacobian $J(x, y) = x_u y_v - x_v y_u$ does not vanish in B . Moreover, for all B_R , $0 < R < 1$, we have

$$\begin{aligned} \|z\|_{C^{1,\mu}(B_R)} &\leq C \\ J(x, y) &\geq c > 0 \end{aligned}$$

where C, c depends on $\mu, \lambda, \Lambda, M_1, M_0, M$ and R .

In order to prove isometric embedding theorem in space form for Dini continuity case, we need to extend the above lemma to the case that a is only Dini continuous. This lemma will be proved in appendix.

Lemma 2.6. *Let $z(w)$ be a homeomorphism from the closed unit disc \bar{B} onto itself of class $W^{1,2}$ solving the system (2.4). Suppose that assumption (4,5) is satisfied, assume that $z(0) = 0$ and*

$$\int_{\Omega} |Dz|^2 dudv \leq M$$

Then $z \in C_{loc}^1(B) \cap C^0(\bar{B})$ and the Jacobian $J(x, y) = x_u y_v - x_v y_u$ does not vanish in B . Moreover, for all B_R , $0 < R < 1$, we have

$$\begin{aligned} \|z\|_{C^1(B_R)} &\leq C \\ J(x, y) &\geq c > 0 \end{aligned}$$

where C, c depends on $\lambda, \Lambda, M, \omega$ and R . In particular, the moduli of continuity of Dz is under control.

3. ISOMETRIC EMBEDDING INTO SPACE FORM

In this section, we prove isometric embedding into space form using Heinz's type interior C^2 estimate, under the condition that $g \in C^2$ with D^2g Dini continuous.

We first state a proposition.

Proposition 3.1. *Let $\Sigma = (\mathbb{S}^2, g)$ be a C^2 surface isometrically embedded into space form N_K^3 , such that the second order derivative is Dini continuous. For any point $(x_0, y_0) \in \Sigma$, let $B_{r_0}(x_0, y_0)$ be a geodesic disk in Σ , where r_0 is a fixed small constant. Suppose that*

$$D = \frac{R - 2K}{2} \geq C_0 > 0$$

$$|g| = \sqrt{\det g_{ij}} \geq M^{-1} > 0, \quad \forall (x, y) \in \bar{B}_{r_0}$$

and

$$\int_{\Sigma} H d\sigma \leq M.$$

Then there exists a homeomorphism $(x, y) = (x(u, v), y(u, v))$ from $\bar{B} = \{u^2 + v^2 \leq 1\}$ onto \bar{B}_{r_0} of class $C_{loc}^1(B) \cap C^0(\bar{B})$ with $x(0) = x_0, y(0) = y_0$ such that

$$(3.1) \quad \begin{aligned} \frac{h_1^1}{\sqrt{D}} &= \frac{y_u^2 + y_v^2}{J(x, y)}, \\ -\frac{h_2^1}{\sqrt{D}} &= \frac{x_u y_u + x_v y_v}{J(x, y)}, \\ \frac{h_2^2}{\sqrt{D}} &= \frac{x_u^2 + x_v^2}{J(x, y)}. \end{aligned}$$

Further more,

$$(3.2) \quad Lx = 0, \quad Ly = 0,$$

where

$$L = \frac{\partial}{\partial u} \left(\sqrt{D} \frac{\partial}{\partial u} \right) + \frac{\partial}{\partial v} \left(\sqrt{D} \frac{\partial}{\partial v} \right).$$

Proof. We first prove the existence of isothermal parameters (u, v) in B_{r_0} .

By (2.3), we have

$$\det(h_j^i) = \frac{R - 2K}{2} = D \geq C_0$$

Thus we can consider

$$ds^2 = h_1^1 dx^2 + 2h_2^1 dx dy + h_2^2 dy^2$$

as a positive metric in B_{r_0} .

Since Σ is a C^2 surface with D^2X Dini continuous, thus h_j^i is Dini continuous. By Lemma 2.1, we have

$$(3.3) \quad ds^2 = \Lambda(du^2 + dv^2), \quad \Lambda \neq 0$$

in each $B_r \subset B_{r_0}$.

This is the local existence of isothermal parameters. We now want to apply uniformization theorem to prove the existence of (u, v) in B_R . To do that, we only need to check the transformation map is holomorphic.

Let $w_1 = (u_1, v_1), w_2 = (u_2, v_2)$ be two isothermal parameters. In the intersection area, we have

$$ds^2 = \Lambda_1(du_1^2 + dv_1^2) = \Lambda_2(du_2^2 + dv_2^2).$$

By simple calculation, we have

$$\begin{aligned} u_{1u_2}^2 + v_{1u_2}^2 &= u_{1v_2}^2 + v_{1v_2}^2, \\ u_{1u_2} u_{1v_2} + v_{1u_2} v_{1v_2} &= 0 \end{aligned}$$

this is in fact the standard Cauchy-Riemann system

$$u_{1u_2} = v_{1v_2}, \quad u_{1v_2} = -v_{1u_1}$$

or

$$w_{1\bar{w}_2} = 0$$

Thus we have a homeomorphism $(x, y) = (x(u, v), y(u, v))$ from $\bar{B} = \{u^2 + v^2 \leq 1\}$ onto \bar{B}_{r_0} with $x(0) = x_0, y(0) = y_0$.

By isothermal parameters (3.3), we have

$$(3.4) \quad \begin{aligned} \sqrt{D}x_u &= h_2^1 x_v + h_2^2 y_v, \\ \sqrt{D}x_v &= -h_2^1 x_u - h_2^2 y_u \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} \sqrt{D}y_u &= -h_1^1 x_v - h_2^1 y_v, \\ \sqrt{D}y_v &= h_1^1 x_u + h_2^1 y_u \end{aligned}$$

The equivalent conformal relations are

$$(3.6) \quad \begin{aligned} \frac{h_1^1}{\sqrt{D}} &= \frac{y_u^2 + y_v^2}{J(x, y)}, \\ -\frac{h_2^1}{\sqrt{D}} &= \frac{x_u y_u + x_v y_v}{J(x, y)}, \\ \frac{h_2^2}{\sqrt{D}} &= \frac{x_u^2 + x_v^2}{J(x, y)}. \end{aligned}$$

We now derive the relation (3.2). By (3.4), we deduce that

$$\begin{aligned} Lx &= (h_2^1)_u x_v - (h_2^1)_v x_u + (h_2^2)_u y_v - (h_2^2)_v y_u \\ &= ((h_2^2)_x - (h_2^1)_y) J(x, y) \end{aligned}$$

By Codazzi equation (2.2)

$$(3.7) \quad \nabla_1 h_2^2 - \nabla_2 h_1^2 = g^{2i} h_{i21} - g^{2i} h_{i12} = 0$$

thus

$$Lx = 0$$

Similarly, we have

$$Ly = 0$$

By (3.6), we have

$$\int_B (|Dx|^2 + |Dy|^2) dudv = \int_{B_{r_0}} \frac{H}{\sqrt{D}} dx dy \leq \frac{1}{\sqrt{C_0} \inf |g|} \int_{\Sigma} H d\sigma \leq C$$

where $|g| = \sqrt{\det g_{ij}}$.

Together with (3.2), Lemma 1.6.3 in [30] and recent theorem in [21] by Y. Li (see Lemma 6.1 below), we conclude that

$$x, y \in C_{loc}^1(B) \cap C^0(\bar{B}).$$

The proposition is now proved. □

We are now in position to prove Theorem 1.3.

Proof. We first need to verify the condition

$$\int_{\Sigma} H d\sigma \leq M$$

Since space form are of warped product structure, we can write the metric \bar{g} as follows

$$\bar{g} = dr^2 + \phi^2(r) g_{\mathbb{S}^2}$$

where $\phi(r) = r$ if $K = 0$, $\phi(r) = \frac{\sinh(\kappa r)}{\kappa}$ if $K = -\kappa^2$, $\phi(r) = \frac{\sin(\kappa r)}{\kappa}$ if $K = \kappa^2$ and $g_{\mathbb{S}^2}$ is the standard metric on \mathbb{S}^2 . Note that $g_{\mathbb{S}^2}$ is different from the metric g on Σ .

Since warped product spaces equip with a conformal Killing vector field, we have the following formulas, see for example in [8].

$$(3.8) \quad \Phi_{ij} = \phi' g_{ij} - h_{ij} \left\langle \phi \frac{\partial}{\partial r}, \nu \right\rangle$$

where $\Phi = \int_0^r \phi(\rho) d\rho$ and ν is the unit outer normal of Σ .

Take trace of (3.8) with σ_2^{ij} , and integrate on Σ , we have

$$\int_{\Sigma} \sigma_2^{ij} \Phi_{ij} d\sigma = \int_{\Sigma} H \phi' d\sigma - 2 \int_{\Sigma} \sigma_2 \left\langle \phi \frac{\partial}{\partial r}, \nu \right\rangle d\sigma$$

By Gauss equation (2.3) and Codazzi equation 2.2, we have

$$\int_{\Sigma} H \phi' d\sigma = \int_{\Sigma} (R + 2\kappa^2) \left\langle \phi \frac{\partial}{\partial r}, \nu \right\rangle d\sigma$$

For $K = 0$, we have $\phi' = 1$. For $K = -\kappa^2$, we have $\phi' = \cosh(\kappa r) \geq 1$. For $K = \kappa^2$, by the Rauch comparison theorem, the diameter of Σ is strictly less than $\frac{\pi}{\kappa}$, by a proper choice of origin, we have $\phi' = \cos(\kappa r) \geq C$.

To sum up, we have

$$(3.9) \quad \int_{\Sigma} H d\sigma \leq \frac{1}{C} \int_{\Sigma} H \phi(\kappa r) d\sigma = \frac{1}{C} \int_{\Sigma} (R + 2\kappa^2) \left\langle \phi \frac{\partial}{\partial r}, \nu \right\rangle d\sigma \leq M$$

as the diameter of Σ is bounded by the $\|g\|_{C^2}$.

We can now apply Lemma 2.6, together with (3.6), we have

$$|h_j^i| \leq C$$

for $z = (x, y) \in B_{r_0/2}$, where C depends only on C_0 , κ , $\|g\|_{C^2}$ and ω .

Moreover, we have for $h_j^i(z) = h_j^i(w)$

$$(3.10) \quad |h_j^i(w_1) - h_j^i(w_2)| \leq C\beta(|w_1 - w_2|)$$

for $z_1, z_2 \in B_{r_0/2}$, where $w_1 = (u_1, v_2), w_2 = (u_2, v_2)$, C and β depends only on $C_0, \kappa, \|g\|_{C^2}$ and ω .

By Proposition 3.1, $x, y \in C^0(\bar{B})$ and map boundary to boundary, this implies there exists $0 < \rho < 1$, such that if $(x, y) \in B_{r_0/2}$, then $u^2 + v^2 < \rho^2$.

By Taylor expansion

$$|z_1 - z_2| = |J(x, y)(\tilde{w})||w_1 - w_2| \geq c|w_1 - w_2|$$

for $z_1, z_2 \in B_{r_0/2}$. Together with (3.10), we have

$$|h_j^i(z_1) - h_j^i(z_2)| \leq C\beta(|z_1 - z_2|)$$

for $z_1, z_2 \in B_{r_0/2}$.

In view of (3.8), we have

$$\|\Phi\|_{C^2} \leq C$$

for $(x, y) \in B_{r_0/2}$, with moduli of continuity of $D^2\Phi$ under control.

The full a priori estimate now follows by the fact that Σ can be covered by finite patches of $B_{r_0/2}$ with $|g| = \sqrt{\det g_{ij}} \geq M^{-1}$.

For the case $K \leq 0$, the theorem follows by Pogorelov's isometric embedding into hyperbolic space. For the case $K > 0$, we can first establish isometric embedding using the openness theorem by Li-Wang [20], i.e. Theorem 5.2 in section 5 and the normalized Ricci flow. The theorem now follows from the a priori estimate established above. □

4. A PRIORI ESTIMATE OF THE EMBEDDING

In this section, we consider the a priori estimate of the embedding. We first state a proposition, which is essentially contained in [15], too. For the sake of completeness, we give the proof here.

Proposition 4.1. *Let $\Sigma = (\mathbb{S}^2, g)$ be a C^3 surface isometrically embedded into a C^3 ambient space (N^3, \bar{g}) , for any point $(x_0, y_0) \in \Sigma$, let $B_{r_0}(x_0, y_0)$ be a geodesic disk in Σ , where r_0 is a fixed small constant. Suppose that*

$$D = \frac{R - \bar{R}}{2} + \bar{Ric}(\nu, \nu) \geq C_0 > 0$$

$$|g| = \sqrt{\det g_{ij}} \geq M^{-1} > 0, \quad \forall (x, y) \in \bar{B}_{r_0}$$

and

$$\int_{\Sigma} H d\mu \leq M.$$

Then there exists a homeomorphism $(x, y) = (x(u, v), y(u, v))$ from $\bar{B} = \{u^2 + v^2 \leq 1\}$ onto \bar{B}_{r_0} of class $C_{loc}^{1,\mu}(B) \cap C^0(\bar{B})$ for any $0 < \mu < 1$ with $x(0) = x_0, y(0) = y_0$ such that

$$(4.1) \quad \begin{aligned} \frac{h_1^1}{\sqrt{D}} &= \frac{y_u^2 + y_v^2}{J(x, y)}, \\ -\frac{h_2^1}{\sqrt{D}} &= \frac{x_u y_u + x_v y_v}{J(x, y)}, \\ \frac{h_2^2}{\sqrt{D}} &= \frac{x_u^2 + x_v^2}{J(x, y)}. \end{aligned}$$

Further more,

$$(4.2) \quad \begin{aligned} \Delta x &= h_1(z)|Dx|^2 + h_2(z)Dx \cdot Dy + h_3(z)|Dy|^2 + h_4(z)Dx \wedge Dy, \\ \Delta y &= \tilde{h}_1(z)|Dx|^2 + \tilde{h}_2(z)Dx \cdot Dy + \tilde{h}_3(z)|Dy|^2 + \tilde{h}_4(z)Dx \wedge Dy, \end{aligned}$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

is the flat Laplacian of (u, v) , and

$$\begin{aligned} h_1 &= -\frac{1}{2D} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha \right), \\ h_2 &= -\frac{1}{2D} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha \right), \\ h_3 &= 0, \\ h_4 &= \frac{g^{2i} \bar{R}_{\nu i 21} - \bar{Ric}(e_1, \nu)}{\sqrt{D}}, \\ \tilde{h}_1 &= 0, \\ \tilde{h}_2 &= -\frac{1}{2D} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha \right), \\ \tilde{h}_3 &= -\frac{1}{2D} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha \right), \\ \tilde{h}_4 &= \frac{g^{1i} \bar{R}_{\nu i 12} - \bar{Ric}(e_2, \nu)}{\sqrt{D}}. \end{aligned}$$

Proof. By (2.3), we have

$$\det(h_j^i) = \frac{R - \bar{R}}{2} + \bar{Ric}(\nu, \nu) = D \geq C_0$$

Thus we can consider

$$ds^2 = h_1^1 dx^2 + 2h_2^1 dx dy + h_2^2 dy^2$$

as a positive metric on B_{r_0} .

Similarly to Proposition 3.1, there exists homeomorphism $(x, y) = (x(u, v), y(u, v))$ from $\bar{B} = \{u^2 + v^2 \leq 1\}$ onto \bar{B}_{r_0} of class $C_{loc}^{1,\mu}(B) \cap C^0(\bar{B})$ for any $0 < \mu < 1$ with $x(0) = x_0, y(0) = y_0$ such that

$$ds^2 = \Lambda(du^2 + dv^2)$$

Note that instead of using the Theorem in [21] by Y. Li, we may use Theorem 2.4.4 in [30] to obtain the $C_{loc}^{1,\mu}$ estimate.

We only need to verify the relation (4.2).

By this isothermal parameters, we have

$$(4.3) \quad \begin{aligned} \sqrt{D}x_u &= h_2^1 x_v + h_2^2 y_v, \\ \sqrt{D}x_v &= -h_2^1 x_u - h_2^2 y_u \end{aligned}$$

and

$$(4.4) \quad \begin{aligned} \sqrt{D}y_u &= -h_1^1 x_v - h_2^1 y_v, \\ \sqrt{D}y_v &= h_1^1 x_u + h_2^1 y_u \end{aligned}$$

By (4.3), we deduce that

$$\begin{aligned} \Delta x &= \left(\frac{h_2^1 x_v + h_2^2 y_v}{\sqrt{D}} \right)_u + \left(\frac{-h_2^1 x_u - h_2^2 y_u}{\sqrt{D}} \right)_v \\ &= \left(\frac{h_2^1}{\sqrt{D}} \right)_y y_u x_v + \left(\frac{h_2^2}{\sqrt{D}} \right)_x x_u y_v - \left(\frac{h_2^1}{\sqrt{D}} \right)_y y_v x_u - \left(\frac{h_2^2}{\sqrt{D}} \right)_x x_v y_u \\ &= \left(\left(\frac{h_2^2}{\sqrt{D}} \right)_x - \left(\frac{h_2^1}{\sqrt{D}} \right)_y \right) J(x, y) \end{aligned}$$

Similarly,

$$\Delta y = \left(\left(\frac{h_1^1}{\sqrt{D}} \right)_y - \left(\frac{h_2^1}{\sqrt{D}} \right)_x \right) J(x, y)$$

Thus

$$(4.5) \quad \Delta x = \frac{\nabla_1 h_2^2 - \nabla_2 h_1^2}{\sqrt{D}} J(x, y) - \frac{h_2^2 D_1 - h_1^2 D_2}{2D^{3/2}} J(x, y)$$

By Codazzi equation (2.2)

$$(4.6) \quad \nabla_1 h_2^2 - \nabla_2 h_1^2 = g^{2i} h_{i21} - g^{2i} h_{i12} = g^{2i} \bar{R}_{\nu i 21}$$

since

$$D = \frac{R - \bar{R}}{2} + \bar{Ric}(\nu, \nu)$$

we have

$$D_i = \frac{R_i - \bar{R}_\alpha X_i^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_i^\alpha + 2\bar{Ric}(e_j, \nu) h_i^j$$

thus

$$\begin{aligned}
(4.7) \quad h_2^2 D_1 - h_1^2 D_2 &= h_2^2 \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha + 2\bar{Ric}(e_j, \nu) h_1^j \right) \\
&\quad - h_1^2 \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha + 2\bar{Ric}(e_j, \nu) h_2^j \right) \\
&= \frac{|Dx|^2 \sqrt{D}}{J(x, y)} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha \right) \\
&\quad + \frac{Dx \cdot Dy \sqrt{D}}{J(x, y)} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha \right) \\
&\quad + 2(h_1^1 h_2^2 - h_2^1 h_1^2) \bar{Ric}(e_1, \nu)
\end{aligned}$$

we have used (4.1) in the second equality.

Plug (4.6) and (4.7) into (4.5), we have

$$\begin{aligned}
h_1 &= -\frac{1}{2D} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha \right), \\
h_2 &= -\frac{1}{2D} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha \right), \\
h_3 &= 0, \\
h_4 &= \frac{g^{2i} \bar{R}_{\nu i 21} - \bar{Ric}(e_1, \nu)}{\sqrt{D}}
\end{aligned}$$

similarly

$$\Delta y = \frac{\nabla_2 h_1^1 - \nabla_1 h_2^1}{\sqrt{D}} J(x, y) - \frac{h_1^1 D_2 - h_2^1 D_1}{2D^{3/2}} J(x, y)$$

$$\nabla_2 h_1^1 - \nabla_1 h_2^1 = g^{1i} h_{i12} - g^{1i} h_{i21} = g^{1i} \bar{R}_{\nu i 12}$$

and

$$\begin{aligned}
h_1^1 D_2 - h_2^1 D_1 &= h_{11} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha + 2\bar{Ric}(e_j, \nu) h_2^j \right) \\
&\quad - h_{12} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha + 2\bar{Ric}(e_j, \nu) h_1^j \right) \\
&= \frac{|Dy|^2 \sqrt{D}}{J(x, y)} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha \right) \\
&\quad + \frac{Dx \cdot Dy \sqrt{D}}{J(x, y)} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha \right) \\
&\quad + 2(h_1^1 h_2^2 - h_2^1 h_1^2) \bar{Ric}(e_2, \nu)
\end{aligned}$$

thus

$$\begin{aligned}\tilde{h}_1 &= 0, \\ \tilde{h}_2 &= -\frac{1}{2D} \left(\frac{R_1 - \bar{R}_\alpha X_1^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_1^\alpha \right), \\ \tilde{h}_3 &= -\frac{1}{2D} \left(\frac{R_2 - \bar{R}_\alpha X_2^\alpha}{2} + \bar{Ric}_\alpha(\nu, \nu) X_2^\alpha \right), \\ \tilde{h}_4 &= \frac{g^{1i} \bar{R}_{\nu i 12} - \bar{Ric}(e_2, \nu)}{\sqrt{D}}.\end{aligned}$$

The proposition is now proved. □

We now prove the bound for principal curvature and it's Hölder norm.

Theorem 4.2. *Let (N^3, \bar{g}) be a 3-dimensional Riemannian manifold with possibly finite many boundary components, which are minimal surfaces. Suppose $\Sigma = (\mathbb{S}^2, g)$ is isometrically embedded into (N^3, \bar{g}) , let X be the embedding, assume that*

$$(4.8) \quad R(x) - \bar{R}(X(x)) + 2\bar{Ric}_{X(x)}(\nu, \nu) \geq C_0 > 0,$$

$\forall x \in \Sigma$. Assume further that $\bar{R} \geq -6\kappa^2$ for some constant $\kappa > 0$. Then we have

$$\|h_j^i\|_{C^\mu} \leq C$$

for any $0 < \mu < 1$, where h_j^i is the Weingarten tensor, C depends on $C_0, \kappa, \|g\|_{C^3}$ and $\|\bar{g}\|_{C^3}$, but not on the position of Σ in N^3 .

Proof. We first verify the condition

$$\int_{\Sigma} H d\sigma \leq M$$

Take κ large enough, then we have

$$R > -2\kappa^2$$

By Lemma 2.3, we have

$$\int_{\Sigma} (H_0 - H) \cosh(\kappa r) d\sigma \geq 0$$

On the other hand, by (3.9) in the proof of Theorem 1.3, we have

$$\int_{\Sigma} H_0 \cosh(\kappa r) d\sigma \leq M$$

where M only depends on κ and $\|g\|_{C^2}$.

Thus

$$\int_{\Sigma} H d\sigma \leq \int_{\Sigma} H \cosh(\kappa r) d\sigma \leq \int_{\Sigma} H_0 \cosh(\kappa r) d\sigma \leq M$$

We can now apply Proposition 4.1, by which we have

$$(4.9) \quad \begin{aligned} \Delta x &= h_1(z)|Dx|^2 + h_2(z)Dx \cdot Dy + h_3(z)|Dy|^2 + h_4(z)Dx \wedge Dy, \\ \Delta y &= \tilde{h}_1(z)|Dx|^2 + \tilde{h}_2(z)Dx \cdot Dy + \tilde{h}_3(z)|Dy|^2 + \tilde{h}_4(z)Dx \wedge Dy, \end{aligned}$$

Moreover, all the assumption of Lemma 2.5 is satisfied, thus Lemma 2.5 is applicable, so we have

$$(4.10) \quad |Dx|^2, |Dy|^2 \leq C, \quad J(x, y) \geq c > 0$$

for $(x, y) \in B_{r_0/2}$.

In view of (4.1), we have

$$|h_j^i| \leq C$$

for $(x, y) \in B_{r_0/2}$.

Similar to the proof of Theorem 1.3, we have

$$(4.11) \quad \|h_j^i\|_{C^\mu} \leq C$$

for $(x, y) \in B_{r_0/2}$.

The theorem now follows by the fact that Σ can be covered by finite patches of $B_{r_0/2}$ with $|g| = \sqrt{\det g_{ij}} \geq M^{-1}$. □

We are now in position to prove Theorem 1.1.

Proof. To begin with, by Theorem 4.2, in order to prove Theorem 1.1, we only need to study the high regularity of the embedding.

Let's restrict ourself in local consideration. First choose O inside Σ and sufficiently close to Σ . Take r such that it is smaller or equal than the injective radius of Σ and N . Let $B_r \subset N$, a geodesic ball centered at O . Denote $S_r = B_r \cap \Sigma$.

Since the second fundamental form of Σ is positive definite, S_r is in fact convex and thus star-shaped w.r.t. O . Moreover, we have $\langle \partial_r, \nu \rangle \geq C_0 > 0$, where ν is the outer unit normal.

Since r is smaller or equal than the injective radius of Σ and N , we can write the metric in B_r as

$$ds^2 = dr^2 + \phi_{ij}(r, \theta)\sigma_{ij}$$

where σ_{ij} is the standard metric on \mathbb{S}^2 , and ϕ_{ij} is a function of r, θ .

Since S_r is star-shaped, we can write S_r as a graph on \mathbb{S}^2 , i.e. (θ, r) . We have

$$e_i = \partial_i + r_i \partial_r$$

and

$$g_{ij} = r_i r_j + \phi_{ij} \sigma_{ij}$$

we then have

$$\nu = \frac{1}{v} \left(-\frac{r^i}{\phi_{ii}} \partial_i + \partial_r \right)$$

where

$$v^2 = 1 + \frac{r_i^2}{\phi_{ii}}$$

Now

$$\begin{aligned} -h_{ij} &= \langle \bar{\nabla}_{e_j} e_i, \nu \rangle \\ &= \frac{1}{v} \left\langle r_j \bar{\nabla}_{\partial_r} (\partial_i + r_i \partial_r) + \bar{\nabla}_{\partial_j} (\partial_i + r_i \partial_r), -\frac{r^k}{\phi_{kk}} \partial_k + \partial_r \right\rangle \end{aligned}$$

since ∂_r is the geodesic vector field, i.e. $\bar{\nabla}_{\partial_r} \partial_r = 0$, we have

$$\begin{aligned} -h_{ij} &= \frac{1}{v} \left\langle r_j \bar{\nabla}_{\partial_r} \partial_i + \bar{\nabla}_{\partial_j} \partial_i + r_i \bar{\nabla}_{\partial_j} \partial_r + r_{ij} \partial_r, -\frac{r^k}{\phi_{kk}} \partial_k + \partial_r \right\rangle \\ &= \frac{1}{v} (r_j \langle \bar{\nabla}_{\partial_r} \partial_i, \partial_r \rangle + \langle \bar{\nabla}_{\partial_j} \partial_i, \partial_r \rangle + r_{ij}) \\ &\quad - \frac{r^k}{v \phi_{kk}} (r_j \langle \bar{\nabla}_{\partial_r} \partial_i, \partial_k \rangle + \langle \bar{\nabla}_{\partial_j} \partial_i, \partial_k \rangle + r_i \langle \bar{\nabla}_{\partial_j} \partial_r, \partial_k \rangle) \\ &= \frac{1}{v} (\langle \bar{\nabla}_{\partial_j} \partial_i, \partial_r \rangle + r_{ij}) - \frac{r^k}{v \phi_{kk}} (-r_j \langle \bar{\nabla}_{\partial_i} \partial_k, \partial_r \rangle + \langle \bar{\nabla}_{\partial_j} \partial_i, \partial_k \rangle - r_i \langle \partial_r, \bar{\nabla}_{\partial_j} \partial_k \rangle) \end{aligned}$$

since

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k (g_{jk,i} + g_{ki,j} - g_{ij,k}) g^{km}$$

we have

$$\Gamma_{ij}^r = -\frac{\partial \phi_{ij}}{\partial r} \frac{\sigma_{ij}}{2}$$

thus

$$\begin{aligned} -h_{ij} &= \frac{1}{v} \left(-\frac{\partial \phi_{ij}}{\partial r} \frac{\sigma_{ij}}{2} + r_{ij} - \frac{r_i r_j}{2 \phi_{ii}} \frac{\partial \phi_{ii}}{\partial r} - \frac{r_i r_j}{2 \phi_{jj}} \frac{\partial \phi_{jj}}{\partial r} - r^k \Gamma_{ij}^k \right) \\ &= \frac{1}{v} \left(r_{i,j} - \frac{\partial \phi_{ij}}{\partial r} \frac{\sigma_{ij}}{2} - \frac{r_i r_j}{2 \phi_{ii}} \frac{\partial \phi_{ii}}{\partial r} - \frac{r_i r_j}{2 \phi_{jj}} \frac{\partial \phi_{jj}}{\partial r} \right) \end{aligned}$$

We first note that $\|r\|_{C^1} \leq C$, which can be deduced by the construction and the metric. By the above representation and Theorem 4.2, we have

$$\|r\|_{C^{2,\mu}} \leq C$$

for any $0 < \mu < 1$.

For high regularity, by Gauss equation, we have

$$\det(h_{ij}) = \det(g_{ij}) \left(\frac{R - \bar{R}}{2} + \bar{Ric}(\nu, \nu) \right)$$

This is a Monge-Ampère type equation with right hand side positive. Since $r \in C^{2,\mu}$, we can apply Schauder estimate to obtain high regularity. Theorem 1.1 is thus proved.

□

5. ISOMETRIC EMBEDDING INTO RIEMANNIAN MANIFOLD

In this section, we prove the isometric embedding into general Riemannian manifold. We use the continuity method to solve the problem. In view of the continuity method, our main result, Theorem 1.1 serves as the closedness part, moreover, since the estimate is independent of the position, it also ensures the uniform ellipticity of the linearized equation. The openness part is done in a recent work by Li-Wang [20].

We now state two theorems by Li-Wang [20].

Theorem 5.1. *Suppose X is a strictly convex surface (\mathbb{S}^2, g) in general Riemannian manifold (N^3, \bar{g}) . Then, for any symmetric two tensor q , there exists some vector field τ satisfying the linearized equation,*

$$(5.1) \quad dX \cdot D\tau = q.$$

Moreover, for τ perpendicular to the kernel of (5.1),

$$\|\tau\|_{C^0} \leq C$$

where C depends on the upper and lower bound of principal curvature and $\|q\|_{C^{1,\alpha}}$, $0 < \alpha < 1$.

Theorem 5.2. *Suppose (\mathbb{S}^2, g) can be isometrically embedded into a general Riemannian manifold (N^3, \bar{g}) as a closed strictly convex surface. Then for any $\alpha \in (0, 1)$, there exists a positive ϵ , depending only on C_0 , $\|g\|_{C^3}$, $\|\tilde{g}\|_{C^3}$, $\|\bar{g}\|_{C^3}$ and α , such that for any smooth metric \tilde{g} on \mathbb{S}^2 satisfying*

$$\|g - \tilde{g}\|_{C^{2,\alpha}} < \epsilon,$$

$(\mathbb{S}^2, \tilde{g})$ can also be isometrically embedded into (N, \bar{g}) as another closed strictly convex surface.

We now prove Theorem 1.2.

Proof. To solve the problem, we indeed need to solve the following equation

$$dX \cdot dX = g$$

By continuity method, we need to solve

$$dX_t \cdot dX_t = g_t$$

for $t \in [0, 1]$. The linearization equation is

$$dX_t \cdot D\tau_t = q_t$$

where τ_t is the variation vector field and D is the connection of (N, \bar{g}) .

The homotopic path g_t consists of three parts. For $t = 0$, we can choose X_0 as a geodesic sphere of sufficient small radius, which is of sufficient large scalar curvature. We choose the normalized Ricci flow of g_0 as the first part of homotopic path as in [22]. At the end of this part, we have a metric $g_{\frac{1}{3}}$ of constant scalar curvature. The third part is the normalized Ricci flow of g , at the end of this part, we have a

metric $g_{\frac{2}{3}}$ of constant scalar curvature. Now we embed both $(\mathbb{S}^2, g_{\frac{1}{3}})$ and $(\mathbb{S}^2, g_{\frac{2}{3}})$ into hyperbolic space. The last part of the homotopic path is to shrink $(\mathbb{S}^2, g_{\frac{2}{3}})$ to $(\mathbb{S}^2, g_{\frac{1}{3}})$ in hyperbolic space. Clearly, from our construction, the condition (1.3) always holds.

Now consider the closedness part, by Theorem 4.2, the mean curvature estimate is independent of the position (\mathbb{S}^2, g) in (N, \bar{g}) , thus the equation system (5.1) is uniformly elliptic, Theorem 5.1 is then applicable, which yields the bound of τ . The uniform C^0 estimate follows by the formula

$$X(x) = X_0(x) + \int_0^1 \tau_t(x) dt$$

C^1 estimate follows from the metric, C^2 and high regularity follows from Theorem 1.1. The openness part follows from Theorem 5.2. This fulfills the continuity method and thus the theorem is proved. \square

6. APPENDIX

To prove Lemma 2.6, we follow the steps in the proof of Theorem 8.3.2 in [30]. Since most of the procedures are the same, we only need to establish three lemmas that are different.

The first one is C^1 estimate for divergence elliptic equations with Dini coefficients. This was recently proved by Y. Li in [21]. The following lemma is only a special case in [21].

Lemma 6.1. *Let $u \in W^{1,2}(B_4)$ be a solution of the following elliptic equation,*

$$D_i(A_{ij}(x)D_j u) = 0, \quad i, j = 1, \dots, n$$

where B_4 is the ball in \mathbb{R}^n of radius 4 at the origin. Assume that A_{ij} is uniformly elliptic and Dini continuous in B_4 . Then $u \in C_{B_1}^1$.

Remark 6.2. *Our assumption is slightly different from Lemma 6.1 above as $a(z, w)$ depends on z, w . However, since a is continuous, we immediately get $w \in C^\mu$ for any $0 < \mu < 1$. Now a can be considered as a function of z , and is still Dini continuous as being Hölder continuous is stronger than being Dini continuous.*

The second one is the replacement of Theorem 7.3.1 in [30], which was again proved by Schulz in [31].

Lemma 6.3. *Let $\varphi(z) \in C^1(\Omega)$ satisfy the elliptic equation*

$$D_i(a(z)D_i \varphi) = 0$$

such that $\lambda \leq a(z, w) \leq \Lambda$ and $a \in L^\infty(\Omega)$. If $\varphi = o(|z|^n)$ as $|z| \rightarrow 0$ for all $n \in \mathbb{N}$, then $\varphi(z) = 0$.

The third lemma is the replacement of Theorem 7.2.1 in [30].

Lemma 6.4. *Let $\varphi(z) \in C^1(\Omega)$ satisfy the elliptic equation*

$$D_i(a(z)D_i \varphi) = 0$$

such that $\lambda \leq a(z, w) \leq \Lambda$ and a is Dini continuous. If $\varphi(z) = o(|z|^n)$ for some $n \in \mathbb{N}_0$, then

$$\lim_{|z| \rightarrow 0, z \neq 0} \frac{\varphi_z(z)}{z^n}$$

exists.

Proof. To begin with, we assume that φ is not identically 0, otherwise there's nothing to prove. Following the steps in [1] and [31], we consider another function satisfying

$$\phi_x = -\frac{a}{a(0)}\varphi_y, \quad \phi_y = \frac{a}{a(0)}\varphi_x$$

Let $\eta = \varphi + i\phi$, we have

$$\eta_{\bar{z}} = \frac{a(0) - a}{a(0) + a} \bar{\eta}_{\bar{z}}$$

By Bers-Nirenberg's representation theorem [2], we have

$$\eta = f(\chi)$$

where f is an analytic function, χ is a homeomorphism satisfying

$$\chi_{\bar{z}} = \frac{a(0) - a}{a(0) + a} \bar{\eta}_{\bar{z}} \chi_z = \frac{a(0) - a}{a(0) + a} \frac{\overline{f_z(\chi)}}{f_z(\chi)} \bar{\chi}_{\bar{z}}$$

with χ, χ^{-1} being Hölder continuous and $\chi(0) = 0$.

Thus we have

$$|\chi(z_1) - \chi(z_2)| \leq C|z_1 - z_2|^\alpha$$

and

$$|\chi(z)| \geq C|z|^{\frac{1}{\alpha}}$$

for some $0 < \alpha < 1$.

Assume that

$$f_z(z) = a_k z^k + a_{k+1} z^{k+1} + \dots$$

where $a_k \neq 0$ and $k \geq 1$. Note that $k \neq 0$, for otherwise η would automatically be a local homeomorphism by the representation above, which is not true by the assumption.

Rewrite the above equation as

$$\chi_{\bar{z}} = A e^{i\theta} \bar{\chi}_{\bar{z}}$$

where $A = \frac{a(0)-a}{a(0)+a}$ and $e^{i\theta} = \frac{\overline{f_z(\chi)}}{f_z(\chi)}$.

We now show that $A e^{i\theta}$ is Dini continuous. To show that, let $z_1 \neq z_2$, if $z_1 = 0$ or $z_2 = 0$, then it's trivial. Suppose now $z_1, z_2 \neq 0$, such that $|z_1| \leq |z_2|$, we split into two cases.

Case 1, $|z_1 - z_2| \leq |z_1|^\beta$, for $\alpha - \frac{1}{\alpha\beta} > 0$. We first show that

$$\left| \frac{\chi(z_2)}{\chi(z_1)} \right| \leq 1 + \left| \frac{\chi(z_2) - \chi(z_1)}{\chi(z_1)} \right| \leq 1 + C \frac{|z_1 - z_2|^\alpha}{|z_1|^{\frac{1}{\alpha}}} \leq 1 + C|z_1 - z_2|^{\alpha - \frac{1}{\alpha\beta}} \leq 2$$

for $|z_1 - z_2|$ sufficiently small.

Now we have

$$\begin{aligned} \left| \frac{\bar{\chi}^k(z_1)}{\chi^k(z_1)} - \frac{\bar{\chi}^k(z_2)}{\chi^k(z_2)} \right| &\leq C \left| \frac{\bar{\chi}^k(z_1) - \bar{\chi}^k(z_2)}{\chi^k(z_1)} \right| + C \left| \frac{\bar{\chi}^k(z_2)(\chi^k(z_2) - \chi^k(z_1))}{\chi^k(z_1)\chi^k(z_2)} \right| \\ &\leq C \frac{|\chi^k(z_1) - \chi^k(z_2)|}{|\chi(z_1)|^k} \leq C \frac{|\chi(z_1) - \chi(z_2)|}{\chi(z_1)} \leq C|z_1 - z_2|^\alpha |z_1|^{-\frac{1}{\alpha}} \leq C|z_1 - z_2|^{\alpha - \frac{1}{\alpha\beta}}. \end{aligned}$$

we have used the fact $|\frac{\chi(z_2)}{\chi(z_1)}| \leq 2$ in the inequality.

Since

$$e^{i\theta} = \frac{\bar{\chi}^k \bar{a}_k + \bar{a}_{k+1} \bar{\chi} + \cdots}{\chi^k a_k + a_{k+1} \chi + \cdots}$$

We have

$$|e^{i\theta}(z_1) - e^{i\theta}(z_2)| \leq C|z_1 - z_2|^\alpha + C|z_1 - z_2|^{\alpha - \frac{1}{\alpha\beta}}.$$

Thus

$$\begin{aligned} |Ae^{i\theta}(z_1) - Ae^{i\theta}(z_2)| &\leq |A(z_1)| |e^{i\theta}(z_1) - e^{i\theta}(z_2)| + |A(z_1) - A(z_2)| |e^{i\theta}(z_2)| \\ &\leq C|z_1 - z_2|^\alpha + C|z_1 - z_2|^{\alpha - \frac{1}{\alpha\beta}} + C\omega(|z_1 - z_2|) \end{aligned}$$

where ω is the moduli of continuity of a .

Case 2, $|z_1 - z_2| \geq |z_1|^\beta$. We have

$$\begin{aligned} |Ae^{i\theta}(z_1) - Ae^{i\theta}(z_2)| &\leq |A(z_1)| |e^{i\theta}(z_1) - e^{i\theta}(z_2)| + |A(z_1) - A(z_2)| |e^{i\theta}(z_2)| \\ &\leq C\omega(|z_1 - z_2|^{\frac{1}{\beta}}) + C\omega(|z_1 - z_2|) \end{aligned}$$

To sum up, we have

$$|Ae^{i\theta}(z_1) - Ae^{i\theta}(z_2)| \leq C|z_1 - z_2|^\alpha + C|z_1 - z_2|^{\alpha - \frac{1}{\alpha\beta}} + C\omega(|z_1 - z_2|^{\frac{1}{\beta}}) + C\omega(|z_1 - z_2|)$$

Let ω_0 be moduli of continuity of $Ae^{i\theta}$ then

$$\omega_0(r) \leq Cr^\alpha + Cr^{\alpha - \frac{1}{\alpha\beta}} + C\omega(r^{\frac{1}{\beta}}) + C\omega(r)$$

since $\omega(r)$ satisfies Dini condition implies that $\omega(r^{\frac{1}{\beta}})$ also satisfies Dini condition, we have $Ae^{i\theta}$ is Dini continuous.

Let $\chi = u + iv$, then we have

$$u_x - v_y + i(v_x + u_y) = Ae^{i\theta}(u_x + v_y + i(u_y - v_x))$$

i.e.

$$\begin{aligned} u_x - v_y &= A \cos \theta (u_x + v_y) - A \sin \theta (u_y - v_x) \\ v_x + u_y &= A \cos \theta (u_y - v_x) + A \sin \theta (u_x + v_y) \end{aligned}$$

equivalently

$$\begin{aligned} (1 - A \cos \theta)u_x + A \sin \theta u_y &= A \sin \theta v_x + (1 + A \cos \theta)v_y \\ -A \sin \theta u_x + (1 - A \cos \theta)u_y &= -(1 + A \cos \theta)v_x + A \sin \theta v_y \end{aligned}$$

thus

$$\begin{aligned}(1 + A^2 - 2A \cos \theta)u_x &= 2A \sin \theta v_x + (1 - A^2)v_y \\ (1 + A^2 - 2A \cos \theta)u_y &= -(1 - A^2)v_x + 2A \sin \theta v_y\end{aligned}$$

i.e.

$$D_i (A_{ij} v_j) = 0$$

where

$$A_{11} = A_{22} = \frac{1 - A^2}{1 + A^2 - 2A \cos \theta}, \quad A_{12} = -A_{21} = \frac{2A \sin \theta}{1 + A^2 - 2A \cos \theta}$$

Since $Ae^{i\theta}$ is Dini continuous, thus A_{ij} is Dini continuous. On the other hand, $A(z)$ is sufficiently small as z tends to 0, thus A_{ij} is uniformly elliptic. Since χ is quasiconformal mapping, $W^{1,2}$ estimate of v follows by classical results (e.g. Lemma 12.1 in [7]). We can now apply Lemma 6.1 to obtain that $v \in C_{loc}^1$.

Similarly,

$$\begin{aligned}(1 + A^2 + 2A \cos \theta)v_x &= 2A \sin \theta u_x - (1 - A^2)v_y \\ (1 + A^2 + 2A \cos \theta)v_y &= (1 - A^2)u_x + 2A \sin \theta v_y\end{aligned}$$

i.e.

$$D_i (B_{ij} u_j) = 0$$

where

$$B_{11} = B_{22} = \frac{1 - A^2}{1 + A^2 + 2A \cos \theta}, \quad A_{12} = -A_{21} = \frac{2A \sin \theta}{1 + A^2 + 2A \cos \theta}$$

and again $u \in C_{loc}^1$.

Thus we conclude that $\chi \in C_{loc}^1$.

Since χ is a homeomorphism and $\chi_{\bar{z}}(0) = 0$, thus we have

$$\chi = C_0 z + o(|z|), \quad \chi_z = C_0 + o(1)$$

for $C_0 \neq 0$. By the representation, the lemma is now proved. □

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